Nonlinearity from geometric interactions: A case example

P. G. Kevrekidis,[1] V. V. Konotop,[2] B. A. Malomed,[3] and A. R. Bishop[4]
[1]*Department of Mathematics and Statistics, Lederle Graduate Research Tower, University of Massachusetts, Amherst, Massachusetts 01003-4515, USA*
[2]*Centro de Física Teórica e Computacional, Complexo Interdisciplinar, Universidade de Lisboa, Avenida Prof. Gama Pinto 2, Lisbon 1649-003, Portugal and Departamento de Física, Universidade de Lisboa, Campo Grande, Ed. C8, Piso 6, Lisboa 1749-016, Portugal*
[3]*Department of Interdisciplinary Studies, Faculty of Engineering, Tel Aviv University, Tel Aviv 69978, Israel*
[4]*Center for Nonlinear Studies and Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA*
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We propose a ladder model wherein dynamical nonlinearity arises from geometry. It includes two strings of particles which are set along rigid rails of a "railroad" and coupled by linear springs. Physical realizations of the model include dust-particle strings in plasma sheaths and chains of microparticles trapped in a strong optical lattice. The transverse couplings between the strings, along with the motion constraint imposed by the rails, generate nonlinearity. It gives rise to robust solitary waves, which are found analytically in the long-wavelength limit, and are obtained in simulations of the full system.

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The importance of the interplay between nonlinear dynamics and geometry in dynamical-lattice settings is well known. The relevant contexts vary from long-range interactions on a fixed curved substrate [1], to substrate-feedback models [2] and coupled atomic chains [3], and from junctions between lattices with different masses [4] to semicircular polymer-like chains [5] and models of bent DNA [6]. The unifying principle in all these situations is that the geometry can significantly affect static configurations of the lattice as well as its dynamical properties (for instance, a variety of outcomes in the interaction of intrinsic localized modes with curvature). In certain cases, the geometry (in particular, a zigzag shape of molecular chains [7]) induces nonlinearity which leads to nontrivial dynamics through competition with material nonlinearity.

The above-mentioned works chiefly aimed to highlight the importance of including the geometry in the dynamical description of the lattice. In particular, in order to understand the complex dynamics in many applications to soft condensed-matter objects, it is necessary to properly account for the interplay of the nonlinearity and lattice discreteness with the curvature [8]. More generally, physical systems at the nanoscale, including nanotubes and electronic waveguide structures [9] are affected by the substrate.

In the present work, we aim to underscore a different point: the geometry not only should be examined in conjunction with nonlinearity, but *geometry alone* may be a source of nonlinearity. In particular, geometric constraints imposed on simple systems whose underlying dynamics is entirely linear readily generate nonlinear interactions. The resulting nonlinearity can sustain stable solitary waves.

As a prototypical example, we consider a fairly simple ladder-type system that we call a "railway model" (RM), see Fig. 1. Our aim is to induce nonlinearity through geometric constraints, while the underlying interactions are purely linear. To this end, we examine the RM shown in Fig. 1, along the two rails of which two arrays of particles are placed, coupled by linear springs. The motion of the particles is restricted along the rail. The nonlinearity is then induced by the linear springs coupling the two chains transversely (Fig. 1).

While our model appears structurally similar to previously considered ladder-type ones [3,10], it is different, as all the underlying interactions are fully linear, there being no nonlinear on-site or inter-site potential. Yet, the geometric constraints induce a nonlinear transverse interaction [11]. In the present work, transverse motion is absent, and the topic of interest is the analysis of longitudinal modes in the generic (nonresonant) case. Accordingly, the continuum approximation amounts to a system of coupled nonlinear Schrödinger (NLS) equations with the cubic nonlinearity, and solitons found in the model are completely different from those predicted in Ref. [3].

The model introduced in this work applies to numerous physical systems. Two very straightforward applications, which correspond to the RM in the literal form, are chains of dust particles in a plasma sheath, and arrays of microparticles trapped in a strong optical lattice (i.e., a periodic potential formed by interference of laser beams). The dust chains in plasmas were studied in detail theoretically [12] and experimentally [13]; moreover, a two-chain configuration was analyzed in Ref. [14]. Arrays of microparticles in the OL field were also created experimentally [15].

In the RM configuration shown in Fig. 1, the displacements and momenta of the particles ("atoms") moving along

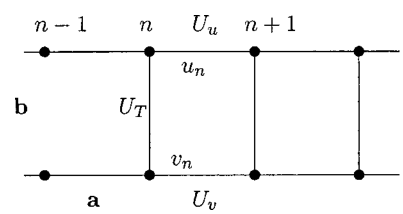

FIG. 1. Schematic representation of the double string sitting on the pair of rails. The upper and lower strings are referred to as u and v chains. Solid lines designate linear interactions in the system.

the two rails are u_n, v_n and p_n, q_n, respectively. The potentials of the linear interaction between the adjacent atoms in the two strings and between them are

$$U_w(w_{n+1}-w_n)=\frac{K_w}{2}\sum_n (w_{n+1}-w_n)^2,\ w=(u,v),$$

$$U_T(u_n-v_n)=\frac{Q}{2}\sum_n (L_n-b)^2. \tag{1}$$

Here $L_n \equiv \sqrt{b^2+(u_n-v_n)^2}$, b is the distance between the rails, and $K_{u,v}$ and Q are the spring constants. The corresponding Hamiltonian is $H=\Sigma_n\{p_n^2/2m_u+q_n^2/2m_v+U_u(u_{n+1}-u_n)+U_v(v_{n+1}-v_n)+U_T(u_n-v_n)\}$, where m_u and m_v are the masses of the particles in each string. It gives rise to the equations of motion

$$m_u\ddot{u}_n=K_u(u_{n+1}+u_{n-1}-2u_n)-Q(u_n-v_n)\times\left[1-\frac{b}{\sqrt{b^2+(u_n-v_n)^2}}\right], \tag{2}$$

$$m_v\ddot{v}_n=K_v(v_{n+1}+v_{n-1}-2v_n)-Q(v_n-u_n)\times\left[1-\frac{b}{\sqrt{b^2+(u_n-v_n)^2}}\right]. \tag{3}$$

Thus the geometry of the problem introduces the nonlinearity in the last terms in Eqs. (2) and (3).

While diatomic systems (with $m_u \neq m_v$) are interesting in their own right, in such contexts as the creation of localized excitations due to the opening of a gap in the linear spectrum [16,17], we hereafter restrict our considerations to the simplest case of the symmetric RM, with $m_u=m_v=m$ and $K_u=K_v=K$. Notice that in the case of the synchronous motion of particles in the two strings, i.e., $u_n=v_n$, the system (2) and (3) is linear indeed, and the solution is obvious, $u_n=A\cos(\omega t+\phi)\sin(kn)$, with $\omega^2=4(K/m)\sin^2(k/2)$.

For more general solutions corresponding to the synchronized motion in the two chains, in the form of $u_n=\alpha v_n$ with constant α, Eqs. (2) and (3) show that the only possibility different from $\alpha=1$ is $\alpha=-1$, i.e., anti-phase motion in the two chains. This is the simplest motion mode in which the geometry-induced nonlinearity manifests itself.

To obtain an approximate analytical form for localized excitations in the RM, we assume that $|u_n-v_n| \ll b$, hence the potentials (1) can be expanded in the Taylor series, so that Eqs. (2) and (3) reduce to ones with the cubic nonlinearity

$$\ddot{u}_n+\frac{K}{m}(2u_n-u_{n+1}-u_{n-1})+\frac{Q}{2mb^2}(u_n-v_n)^3=0,$$

$$\ddot{v}_n+\frac{K}{m}(2v_n-v_{n+1}-v_{n-1})+\frac{Q}{2mb^2}(v_n-u_n)^3=0. \tag{4}$$

Next, the lattice displacements are looked for in the form (see, e.g., Ref. [16]) $u_n(t)=\epsilon[u(t_2,\ldots;x_1,\ldots)(-1)^n e^{-i\omega t_0}+O(\epsilon)]+\text{c.c.}$, $v_n(t)=\epsilon[v(t_2,\ldots;x_1,\ldots)(-1)^n e^{-i\omega t_0}+O(\epsilon)]+\text{c.c.}$, c.c. Here we assume that the motion is *staggered*, which is accounted for by the factor $(-1)^n$, and introduce a hierarchy of stretched temporal and spatial scales, $t_j \equiv \epsilon^j t$ and $n_j \equiv \epsilon^j n$ $(j=0,1,2,\ldots)$, assuming that the functions u and v are continuous functions of the latter variables. Then, n_j are replaced by stretched *continuum* variables, $x_j \equiv a n_j (j \geqslant 1)$. Only the most rapid scales will be explicitly indicated as arguments of the functions.

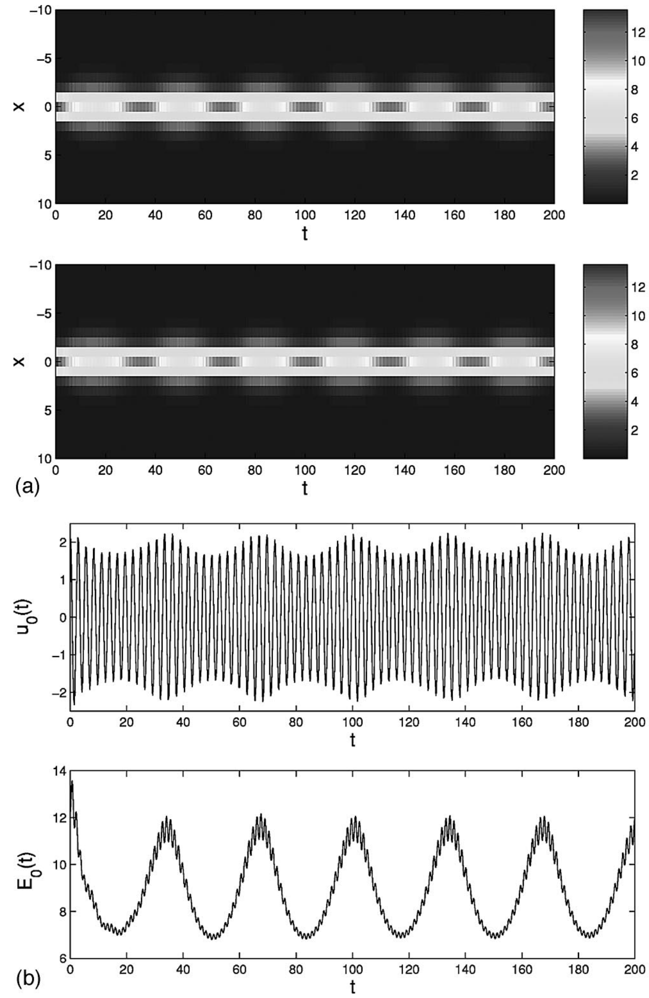

FIG. 2. Panel (a) shows the spatio-temporal evolution of the energy density in the two chains by means of contour plots (top and bottom parts pertain to the u and v chains). The initial condition was taken as per the continuum-limit soliton. The solution retains its out-of-phase character ($u_n=-v_n$) at all $t>0$. Panel (b) shows the motion of the central particle in the u chain (top), and its energy (bottom).

Through straightforward calculations, we thus find $\omega=2\sqrt{K}$ and arrive at the lowest-order equations

$$i\frac{\partial u}{\partial t_2}=\frac{\omega}{8}\frac{\partial^2 u}{\partial x_1^2}+\frac{3Q}{4b^2\omega}(|u|^2u-|v|^2v+u*v^2-v*u^2+2u|v|^2-2v|u|^2), \tag{5}$$

$$i\frac{\partial v}{\partial t_2}=\frac{\omega}{8}\frac{\partial^2 v}{\partial x_1^2}-\frac{3Q}{4b^2\omega}(|u|^2u-|v|^2v+u*v^2-v*u^2+2u|v|^2-2v|u|^2). \tag{6}$$

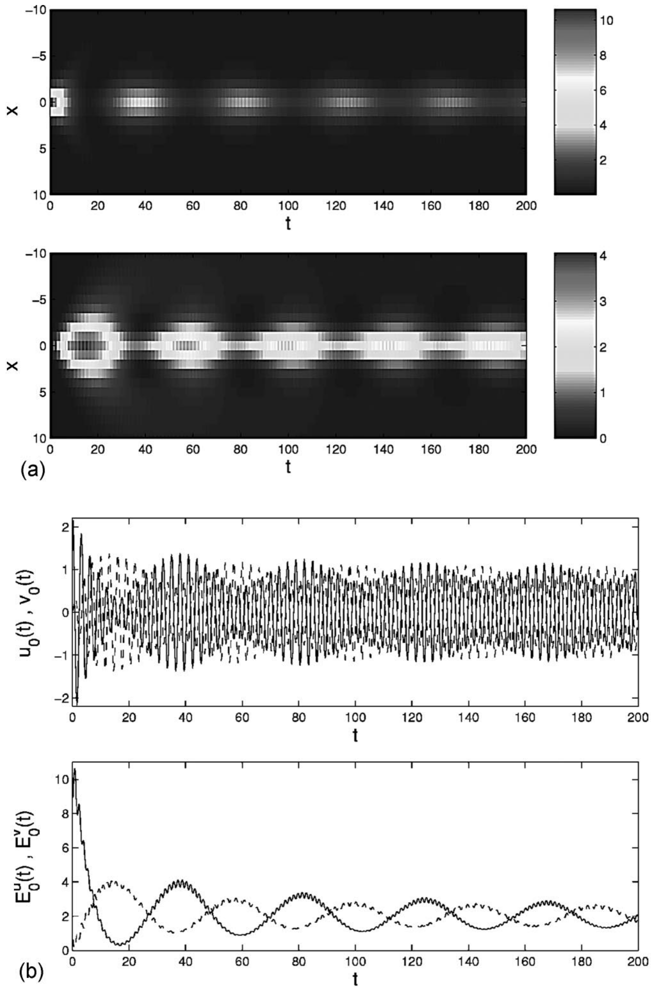

FIG. 3. The same as above, but with the zero initial field in the v chain. The motion and energy of the central particle in the v chain are also shown (by dashed lines) in the panel (b).

As mentioned above, the simplest nontrivial solutions can be sought in the form $u=-v\equiv(2\sqrt{6Q}/b\omega)\psi$. Then Eqs. (6) and (5) reduce to a single NLS equation $i(\partial\psi/\partial\tau)=\partial^2\psi/\partial x_1^2+2|\psi|^2\psi$ with $\tau\equiv\omega t_2/8$. One can thus construct approximate solutions to Eqs. (4), using the $u_n(t)$ and $v_n(t)$ expansions and the solutions of the single NLS equation as the lowest-order approximation. In particular, one can take the NLS soliton $\psi=\eta\,\mathrm{sech}(\eta x_1)\exp(-i\eta^2\tau)$, or either of the following cnoidal (periodic) waves (q is the elliptic function modulus): $\psi=(\eta_1+\eta_2/2)\mathrm{dn}[(\eta_1+\eta_2/2)x_1,q)e^{-i/2(\eta_1^2+\eta_2^2)\tau}$, where $q^2=4\eta_1\eta_2/(\eta_1+\eta_2)^2$, and $\psi=\eta\,\mathrm{cn}(\sqrt{\eta^2+\xi^2}x_1,q)e^{i(\xi^2-\eta^2)\tau}$, where $q^2=\eta^2/(\eta^2+\xi^2)$.

To verify the existence and stability of the nonlinear excitations predicted above in the small-amplitude quasi-continuum approximation, we performed direct simulations of Eqs. (2) and (3) with $m_u=m_v$ and $K_u=K_v$. We looked for nonlinear excitations, similar to those predicted above, not only in the small-amplitude case, but also far from that limit (in particular, when the displacements of the atoms are comparable to the separation b between the rails). Below, we demonstrate nonlinear excitations whose intrinsic spatial scale is definitely much smaller than the size of the integration domain, so that the dynamical behavior is not affected by boundary conditions.

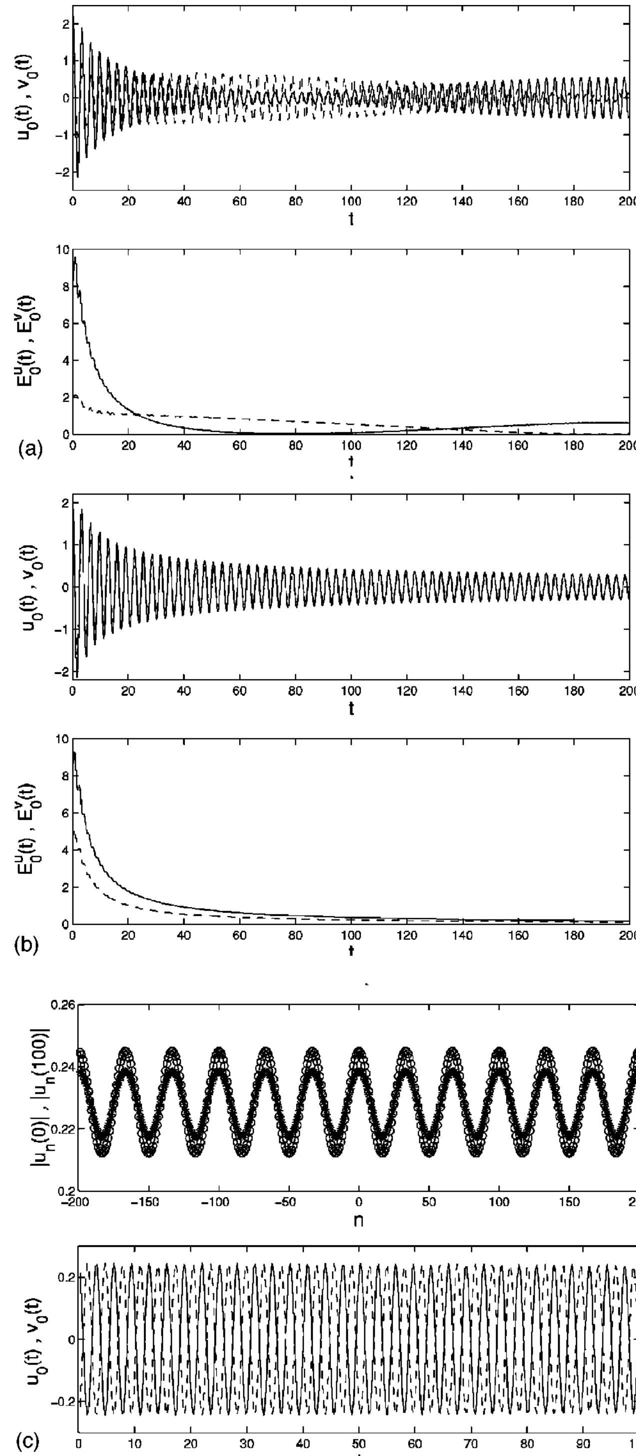

FIG. 4. Panel (a) shows the same as panel (b) in the previous figures for the initial configuration with $v_n=0.5u_n$, while panel (b) shows the case of the initial condition with $v_n=0.75u_n$. Panel (c) shows $|u_n|$ for $t=0$ (circles) and $t=100$ (stars) for the solution excited as per the quasi-continuum cnoidal pattern.The lower subplot shows the motion of the central particles, $u_0(t)=-v_0(t)$, for the latter solution.

Results are displayed in Figs. 2–4. Typical parameter values used were $Q/m=1$ (recall that we have scaled $K/m\equiv1$), and $b=1$. In Fig. 2 the soliton (with the phase shift of π between the two chains) corresponding to the NLS soliton was used as the initial condition. We mostly focused on the solitons rather than periodic waves, as the latter seem less straightforward to excite in the experiment, while the soliton can be easily excited by setting a few atoms in motion. Obviously, the initial configuration readily excites a robust solitary-wave structure, which persists without significant loss.

The deviation of the initial condition from the exact lattice-soliton solution results in excitation of an internal mode in the soliton, which also persists in the simulations. Among two clearly observed frequencies of the displacement pattern, the larger one is the overall frequency of the soliton, while the smaller one belongs to its internal mode.

It is worth noting that, even in the numerical experiment in which only the u chain is initially excited, by creating the continuum-limit soliton component in it (Fig. 3), while the v chain is initially at rest, the anti-phase motion of the coupled chains is still an effective attractor, i.e., the chains readily lock themselves into the out-of-phase mode. This demonstrates the robustness of the nonlinear behavior, and its relevance to a large set of initial conditions (including, in particular, those of the form $v_n=-\alpha u_n$, with any positive α).

In Fig. 4, we report numerical experiments with initial conditions that were even farther from the anti-phase state. In particular, we initialized the motion by setting $v_n=0.5u_n$ [panel (a)] and $v_n=0.75u_n$ [panel (b)], which seems much closer to the in-phase mode, that gives rise to the purely linear motion (see above), than to the nonlinear anti-phase mode. We observe that in the former case, the system still tends to lock into an out-of-phase configuration (but it requires a long relaxation time to do it). In the latter case, when the initial configuration was very close to the in-phase state, the system did eventually lock into the the linear in-phase motion. However, the results clearly demonstrate the robustness of the nonlinear behavior of the solitary waves in the RM.

In panel (c) of Fig. 4, we also show an example of the evolution of a spatially periodic pattern, initialized by the continuum-limit solution. As is seen, such solutions also readjust their profiles, and persist for long times.

We have proposed a model which exemplifies nonlinearity emerging from linear interactions through a geometric restriction imposed on the motion in a simple ladder-shaped system of particles with the interactions between them mediated by linear springs. The model finds its physical realizations in terms of dust chains in plasmas, or arrays of microparticles trapped in strong optical lattices. The resulting form of the geometrically induced nonlinearity was derived and investigated. Remarkable robustness of the corresponding lattice solitons was demonstrated numerically.

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